

Rotationally invariant bipartite states and bound entanglement

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We consider rotationally invariant states in $\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ Hilbert space with even $N_1 \geq 4$ and arbitrary $N_2 \geq N_1$, and show that in such case there always exist states which are inseparable and remain positive after partial transposition, and thus the PPT criterion does not suffice to prove separability of such systems. We demonstrate it applying a map developed recently by Breuer [H.-P. Breuer, Phys. Rev. Lett **97**, 080501 (2006)] to states that remain invariant after partial time reversal.

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I. INTRODUCTION

One of the most important problems of rapidly developing branch of science Quantum Information Theory [1] is to determine whether a given quantum state is separable or entangled. We say that a given state ϱ acting on a finite dimensional product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is separable or classically correlated if it can be written as a convex linear combination of product density operators [2], i.e.,

$$\varrho = \sum_n p_n \varrho_n^{(1)} \otimes \varrho_n^{(2)}, \quad (1)$$

where $\varrho_n^{(1(2))} \in \mathcal{B}(\mathcal{H}_{1(2)})$ for all n and p_n are nonnegative coefficients fulfilling the condition $\sum_n p_n = 1$. Otherwise the state is called entangled or inseparable. An important necessary and sufficient criterion used to check the separability of a state was developed by the Horodecki's [3]. It applies the positive but not completely positive maps, precisely, it states that a density matrix ϱ is separable if and only if the operator $(I \otimes \Lambda)(\varrho)$ is positive for all positive but not completely positive maps $\Lambda : \mathcal{B}(\mathcal{H}_2) \rightarrow \mathcal{B}(\mathcal{H}_1)$. Even though making use of the criterion is not an easy task, together with the result of Peres [4] it provides a strong condition for separability, based on partial transposition. It says that ϱ is entangled if it has a nonpositive partial transposition. For low dimensional systems such as $2 \otimes 2$ and $2 \otimes 3$ this is also a sufficient criterion for separability [3], however, in general there exist states that have positive partial transposition (PPT) and simultaneously are entangled [5]. Such operators are certainly nondistillable [6] and belong to the class of bound entangled (BE) states.

An interesting question is thus whether there exist classes of states (except for $2 \otimes 2$ and $2 \otimes 3$ systems) where partial transposition provides a necessary and sufficient test for separability. One could expect that invariance of states under certain group of symmetry would lead to some interesting results. Such states have a relatively simple structure and therefore have been studied extensively in the literature [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. In particular, it was shown recently that positive partial transposition is a sufficient criterion in the case of $2 \otimes N$ rotationally invariant states [11, 12] and $3 \otimes N$ rotationally invariant states with integer total angular momentum [13, 14]. On the other hand in Ref. [18] it was shown that for $N \otimes N$ rotationally invariant systems with even $N \geq 4$ there always exist bound entangled states detected by certain map.

It is the purpose of the present paper to consider the separability of more general $SO(3)$ invariant states. We concentrate on $N_1 \otimes N_2$ systems with even $N_1 \geq 4$ and arbitrary $N_2 \geq N_1$ and show that there is always a region in the PPT set where states are bound entangled proving, thus, that in such systems positive partial transposition is only a necessary condition for separability. We achieve this using a recently introduced positive indecomposable map Φ given by Eq.(2) [18]. The map belongs to the class of indecomposable positive maps arising from the reduction criterion [19, 20] which was studied in detail by Hall [21]. The action of Φ on a given operator B from $\mathcal{B}(\mathbb{C}^N)$ is such that

$$\Phi(B) = (\text{Tr} B) \mathbb{1}_N - B - \vartheta(B), \quad (2)$$

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where ϑ denotes the time reversal operation and $\mathbb{1}_N$ is a $N \times N$ identity matrix. The map Φ is positive if N is an even number and therefore leads to the following necessary condition for separability on $\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ Hilbert space with even N_1 :

$$\Phi_1(\varrho) \equiv (\Phi \otimes I)(\varrho) \geq 0. \quad (3)$$

We show that when applying the Breuer's map to rotationally invariant states one can restrict to a family of operators invariant under partial time reversal. Moreover the results obtained for ϑ_1 -invariant states can be extended to the PPT rotationally invariant states. We also show that in our case the set of states invariant under partial time reversal can be easily found and therefore it is possible to prove the existence of BE states in higher dimensional rotationally invariant systems.

The paper is organized as follows. In Sec. II we give a brief description of representations of $SO(3)$ -invariant states and action of certain positive but not completely positive maps on this class of states. In Sec. III we present a special case of $4 \otimes N$ and on the basis of this example analyze more general case $N_1 \otimes N_2$ with even $N_1 \geq 4$ and arbitrary $N_2 \geq N_1$. In particular we show that to determine the set of BE states detected by the Breuer's map one can restrict to the subset of states invariant under partial time reversal and then extend the result to the set of PPT states. The results also suggest that the map (2) could provide both necessary and sufficient separability criterion for states invariant under partial time reversal.

II. ROTATIONALLY INVARIANT STATES

A. Representations

Assume that we are given a bipartite quantum state represented by a density matrix ϱ acting on a finite product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 = \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ such that $N_1 \leq N_2$. The angular momenta of the particles are $j_1 = (N_1 - 1)/2$ and $j_2 = (N_2 - 1)/2$, respectively. The characterization of entanglement of such states is in general a difficult task. However, it simplifies when one impose some constraints on considered density matrices. Hereafter we shall be assuming that ϱ is invariant under the action of $SO(3)$ group. More rigorously this assumption means that the following relation holds

$$\left[\mathcal{D}^{(j_1)}(R) \otimes \mathcal{D}^{(j_2)}(R), \varrho \right] = 0 \quad (4)$$

for all proper rotations R from $SO(3)$ group. Here $\mathcal{D}^{(j_1(j_2))}(R)$ denote the unitary irreducible representation of the $SO(3)$ group on the respective state spaces. In view of the Shur's lemma, the states which obey the above condition, can be written in the following form

$$\varrho = \frac{1}{\sqrt{N_1 N_2}} \sum_{J=|j_1-j_2|}^{j_1+j_2} \frac{\alpha_J}{\sqrt{2J+1}} P_J, \quad P_J = \sum_{M=-J}^J |JM\rangle \langle JM|, \quad (5)$$

where $|JM\rangle$ are the common eigenvectors of the square of total angular momentum operator and of its z -component. Thus the set of rotationally invariant states is isomorphic to a proper subset of vectors α from \mathbb{R}^{N_1} , which are nonnegative, i.e., $\alpha_J \geq 0$ and fulfil the following normalization condition

$$\sum_{J=|j_1-j_2|}^{j_1+j_2} \sqrt{\frac{2J+1}{N_1 N_2}} \alpha_J = 1. \quad (6)$$

On the other hand, as proposed by Breuer in Refs. [13, 14], each rotationally invariant state can be written as a combination of Hermitian operators

$$Q_K = \sum_{q=-K}^K T_{K,q}^{(1)} \otimes T_{K,q}^{(2)\dagger}, \quad K = 0, 1, \dots, 2j_1, \quad (7)$$

where $T_{K,q}^{(i)}$ are the components of an irreducible tensor operator [22]. Subscript K is the rank of this tensor operator and q takes on $2K+1$ values, $q = -K, \dots, K$. The operators Q_K are rotationally invariant and form a complete set in

the space of rotationally invariant states. Thus any $SO(3)$ invariant state can be written as their linear combination, i.e.,

$$\varrho = \frac{1}{\sqrt{N_1 N_2}} \sum_{K=0}^{2j_1} \frac{\beta_K}{\sqrt{2K+1}} Q_K. \quad (8)$$

Like in the case of P_J representation the state is uniquely characterized by a parameter vector $\beta \in \mathbb{R}^{N_1}$. Due to the fact that tensor operators Q_K are traceless for $K \neq 0$ the normalization condition gives $\beta_0 = 1$. As it was shown in Ref. [14], the N_1 -dimensional vectors α and β are related by a linear transformation $\beta = L\alpha$ with matrix elements of the orthogonal matrix L given by

$$L_{KJ} = \sqrt{(2K+1)(2J+1)} (-1)^{j_1+j_2+J} \begin{Bmatrix} j_1 & j_2 & J \\ j_2 & j_1 & K \end{Bmatrix}. \quad (9)$$

The reason for introducing two different representations is their convenience for certain purposes. P_J -representation is suitable to determine the state space, whereas the operation of partial time reversal, unitarily equivalent to the partial transposition, can be performed much more easily in the Q_K basis. Thus we apply the latter to identify the set of rotationally invariant states with positive partial time reversal which is the same as set of states that remain positive after partial transposition.

B. Separability

To determine the set of rotationally invariant PPT states (from now on denoted by \mathcal{R}_{ppt}) we apply a map unitarily equivalent to partial transposition, i.e., the partial time reversal map [14]. It is more useful in our case due to the fact that, unlike the partial transposition, it preserves the $SO(3)$ invariance of the state. The operation of partial time reversal acts as follows

$$\vartheta_1(B) = (\vartheta \otimes I)(B), \quad (10)$$

where ϑ is the time reversal map acting on a given operator B as $\vartheta(B) = V B^T V^\dagger$. Here V is a unitary rotation by the angle π about the y -axis. As mentioned before, the action of partial time reversal operator is especially simple in the Q_K representation

$$\vartheta_1 : \beta_K \rightarrow (-1)^K \beta_K, \quad (11)$$

which follows directly from the relation $\vartheta_1(Q_K) = (-1)^K Q_K$ and equation (8). Let us recall one more property of ϑ_1 map, i.e., preservation of separability. It means that if ϱ is separable then $\vartheta_1(\varrho)$ is also separable. We apply this fact in the next section to prove the separability of certain PPT states.

Since the set of separable states is a subset of PPT states, one should have the condition unambiguously determining the separability of analyzed states. A method useful to identify the separable invariant states was developed in [7] and is based on the action of a projection super-operator Π . The super-operator for $SO(3)$ group of symmetry, considered in this paper, projects each state onto a rotationally invariant state space as follows

$$\Pi(\varrho) = \sum_{J=|j_1-j_2|}^{j_1+j_2} \frac{\text{Tr}(P_J \varrho)}{2J+1} P_J = \sum_{K=0}^{2j_1} \frac{\text{Tr}(Q_K \varrho)}{2K+1} Q_K, \quad (12)$$

preserving the separability of a state. The last property allows us to identify the set of separable states \mathcal{R}_{sep} with a convex linear combination of Π -projections of pure normalized product states $|\phi^{(1)}\phi^{(2)}\rangle \in \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$:

$$\mathcal{R}_{\text{sep}} = \text{conv} \left\{ \varrho : \varrho = \Pi(P_{\phi^{(1)}} \otimes P_{\phi^{(2)}}) \right\}, \quad (13)$$

where $P_{\phi^{(i)}}$ is the projection onto the state $|\phi^{(i)}\rangle$. Consequently to find separable states among PPT states it is enough to show that for the extreme points of \mathcal{R}_{sep} (denoted by ϱ_{ext}) there exist pure normalized product states satisfying relation $\Pi(P_{\phi^{(1)}} \otimes P_{\phi^{(2)}}) = \varrho_{\text{ext}}$. However to find the "suspected" extreme points ϱ_{ext} one should have a criterion detecting at least some bound entanglement, so that one could look for separable states in a smaller set.

In our case the map developed recently in [18] proved to be especially useful. It leads to the necessary criterion (3). Using this criterion we show that the set of PPT rotationally invariant states for even $N_1 \leq N_2$ always contains entangled states. To prove this it is enough to check the criterion on the subset of ϑ_1 -invariant states

$$\varrho_{\text{inv}} = \frac{1}{2}[\varrho + \vartheta_1(\varrho)], \quad \varrho \in \mathcal{R}_{\text{ppt}}, \quad (14)$$

since the action of Breuer's map is the same for all states ϱ satisfying above equation and equivalent to

$$\Phi_1(\varrho) = \mathbb{1}_{N_1} \otimes \text{Tr}_1(\varrho) - \varrho - \vartheta_1(\varrho) = \frac{1}{N_2} \mathbb{1}_{N_1} \otimes \mathbb{1}_{N_2} - 2\varrho_{\text{inv}}. \quad (15)$$

In the second equality appearing in Eq. (15) we make use of the fact that $\text{Tr}_1(\varrho) = (1/N_2)\mathbb{1}_{N_2}$ for all rotationally invariant states. As a result of the above relations we can simplify the criterion (3) to

$$\frac{1}{N_2} \mathbb{1}_{N_1} \otimes \mathbb{1}_{N_2} - 2\varrho_{\text{inv}} \geq 0. \quad (16)$$

It automatically follows that if ϱ_{inv} fulfils the criterion (16) then all ϱ 's from equation (14) satisfy (3).

III. ROTATIONALLY INVARIANT STATES AND BOUND ENTANGLEMENT

In this section we shall consider rotationally invariant states in the context of bound entanglement. We show that in $\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ Hilbert space with even N_1 and arbitrary $N_2 \geq N_1$ there always exist PPT entangled states.

At the very beginning to provide some insight into the state space structure we focus on the case of the $4 \otimes N$ system, since it may be easily visualized in \mathbb{R}^3 .

A. $4 \otimes N$ system

Let us now restrict our attention to the case of $N_1 = 4$ ($j_1 = 3/2$) and arbitrary $N_2 = 2j_2 + 1 \geq N_1$ from now on denoted by N . In this paragraph we will also denote j_2 by j . The total angular momentum of the system J takes on the values $J = j - 3/2, \dots, j + 3/2$ and thus every rotationally invariant state is represented by 4 coordinates which satisfy the conditions $\alpha_J \geq 0$ and

$$\sum_{J=j-3/2}^{j+3/2} \sqrt{\frac{2J+1}{4N}} \alpha_J = 1. \quad (17)$$

By the normalization condition the number of independent parameters α_J describing a state can be reduced to 3. In order to characterize this set of rotationally invariant states we need to find its extreme points in α space. Straightforward calculations lead to

$$A_\alpha = \left(\sqrt{\frac{4N}{N-3}}, 0, 0, 0 \right), \quad B_\alpha = \left(0, \sqrt{\frac{4N}{N-1}}, 0, 0 \right), \quad C_\alpha = \left(0, 0, \sqrt{\frac{4N}{N+1}}, 0 \right), \quad D_\alpha = \left(0, 0, 0, \sqrt{\frac{4N}{N+3}} \right).$$

To determine the set of PPT states we transform the vectors α to β using the linear transformation L which matrix elements are given by Eq. (9). In the considered case of $4 \otimes N$ the transformation matrix is of the form

$$L = \frac{1}{2} \begin{pmatrix} \sqrt{\frac{N-3}{N}} & \sqrt{\frac{N-1}{N}} & \sqrt{\frac{N+1}{N}} & \sqrt{\frac{N+3}{N}} \\ -3\sqrt{\frac{(N-3)(N+1)}{5(N-1)N}} & -\frac{N+7}{\sqrt{5N(N+1)}} & \frac{N+7}{\sqrt{5N(N-1)}} & -3\sqrt{\frac{(N+3)(N-1)}{5(N+1)N}} \\ \sqrt{\frac{(N-3)(N+1)(N+2)}{N(N-1)(N-2)}} & \frac{(N-5)\sqrt{N+2}}{\sqrt{(N-2)N(N+1)}} & -\frac{(N+5)\sqrt{N-2}}{\sqrt{(N-1)N(N+2)}} & \sqrt{\frac{(N-1)(N-2)(N+3)}{N(N+1)(N+2)}} \\ -\sqrt{\frac{(N+1)(N+2)(N+3)}{5(N-2)(N-1)N}} & 3\sqrt{\frac{(N^2-9)(N+2)}{(N-2)N(N+1)}} & -3\sqrt{\frac{(N^2-9)(N-2)}{(N-1)N(N+2)}} & \sqrt{\frac{(N-3)(N-2)(N-1)}{5N(N+1)(N+2)}} \end{pmatrix}. \quad (18)$$

At the same time we observe, that the 4-dimensional vectors β are unambiguously characterized by three coordinates since $\beta_0 = 1$ for all rotationally invariant density matrices. This allows us to restrict our considerations to the three parameters $\beta_1, \beta_2, \beta_3$ and visualize all considered sets in \mathbb{R}^3 . The L transformation carried on the extreme points gives us the vertices of the tetrahedron in the space of parameters $\beta_1, \beta_2, \beta_3$ as follows

$$A = \left(-3\sqrt{\frac{N+1}{N-1}}, \sqrt{\frac{(N+1)(N+2)}{(N-1)(N-2)}}, \sqrt{\frac{(N+1)(N+2)(N+3)}{5(N-1)(N-2)(N-3)}} \right),$$

$$B = \left(-\frac{N+7}{N+1}\sqrt{\frac{N+1}{5(N-1)}}, -\frac{N-5}{N+1}\sqrt{\frac{(N+1)(N+2)}{(N-1)(N-2)}}, 3\sqrt{\frac{(N+2)(N+3)(N-3)}{5(N-2)(N-1)(N+1)}} \right),$$

$$C = \left(\frac{N-7}{N+1} \sqrt{\frac{N+1}{5(N-1)}}, -\frac{N+5}{N-1} \sqrt{\frac{(N-1)(N-2)}{(N+1)(N+2)}}, -3 \sqrt{\frac{(N-2)(N-3)(N+3)}{5(N+2)(N-1)(N+1)}} \right),$$

$$D = \left(3 \sqrt{\frac{N-1}{5(N+1)}}, \sqrt{\frac{(N-2)(N-1)}{(N+2)(N+1)}}, \sqrt{\frac{(N-1)(N-2)(N-3)}{5(N+1)(N+2)(N+3)}} \right).$$

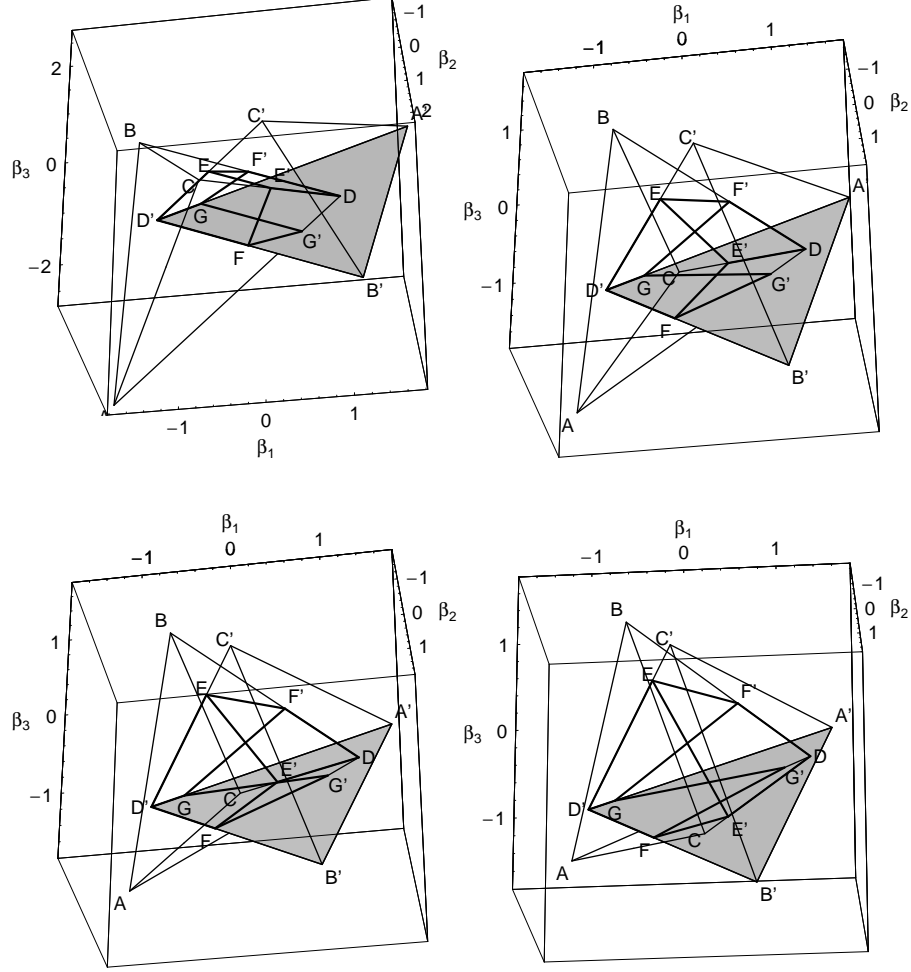


FIG. 1: The set of rotationally invariant states in $\mathbb{C}^4 \otimes \mathbb{C}^N$ (tetrahedron $ABCD$), its image under partial time reversal ($A'B'C'D'$), and intersection of the sets ($DD'EE'FF'GG'$) for various values of j , namely $j = 3/2$ (left upper), $j = 5/2$ (right upper), $j = 7/2$ (left lower), $j = 11/2$ (right lower). The PPT region becomes larger with the growth of dimension of the second subsystem and the PPT criterion fails to detect many entangled states with the growth of asymmetry between the dimensions of subsystems.

Now we can find the image of the tetrahedron $ABCD$ under the action of ϑ_1 . The extreme points of this set transform according to the relation (11), consequently only the sign of β_1 and β_3 coordinates change. We denote the points corresponding to A, B, C, D by A', B', C' and D' , respectively. To identify the set of PPT states we need to find the intersection of the tetrahedrons $ABCD$ and $A'B'C'D'$. Straightforward calculations lead to the set $DD'EE'FF'GG'$ (see Fig. 1). One may easily verify that the pairs of points $(E, E'), (F, F'), (G, G')$ are the images of each other under ϑ_1 (e.g. $E' = \vartheta_1(E)$) and thus it is sufficient to find coordinates of the points E, F , and G , which are as follows

$$\begin{aligned}
E &= \left(-\sqrt{\frac{N-1}{5(N+1)}}, -\sqrt{\frac{(N-1)(N-2)}{(N+1)(N+2)}}, 3\sqrt{\frac{(N-1)(N-2)(N-3)}{5(N+1)(N+2)(N+3)}} \right), \\
F &= \left(-\frac{9(N-4)}{7N-20}\sqrt{\frac{N-1}{5(N+1)}}, \frac{N+4}{7N-20}\sqrt{\frac{(N-1)(N-2)}{(N+1)(N+2)}}, -\frac{13N+28}{7N-20}\sqrt{\frac{(N-1)(N-2)(N-3)}{5(N+1)(N+2)(N+3)}} \right), \\
G &= \left(-\frac{3(N+1)}{N+5}\sqrt{\frac{N-1}{5(N+1)}}, \frac{N+2}{N+5}\sqrt{\frac{(N-1)(N+1)}{(N-2)(N+2)}}, -\frac{N-7}{N+5}\sqrt{\frac{(N-1)(N+1)(N+2)}{5(N-3)(N+3)(N-2)}} \right).
\end{aligned}$$

The described sets are presented in Fig. 1 for $N = 4, 6, 8, 12$. One could see that the overlap of tetrahedrons $ABCD$ and $A'B'C'D'$ grows with the increase of N . This immediately leads to the conclusion that the NPT set shrinks with increasing N . Now applying the methods described in the previous section we characterize the separability of PPT states.

Firstly we apply the Breuer's criterion to the subset of ϑ_1 -invariant states, namely states represented by points lying on the line $E''G''$ (see Fig. 2)

$$\varrho_{\text{inv}}(t) = (1-t)E'' + tG'', \quad t \in [0, 1]. \quad (19)$$

The criterion (15) applied to $\varrho_{\text{inv}}(t)$ leads to the operator with $\alpha_{j-3/2}$ given by:

$$\alpha_{j-3/2}(t) = \sqrt{\frac{N-3}{N}} \left(1 - \frac{(N-1)(N+4)}{(N-2)(N+5)}t \right). \quad (20)$$

As it may be easily verified that remaining α_J 's are nonnegative for all values of t and thus the nonpositivity condition reduces to

$$1 - \frac{(N-1)(N+4)}{(N-2)(N+5)}t < 0. \quad (21)$$

States $\varrho(t)$ for t satisfying the inequality are entangled. The parameter t for which the LHS of (21) equals zero represents the point D'' lying in the middle of the line DD' . At the same time the inequality is satisfied by $t = 1$ which implies that points between D'' and G'' , including G'' represent entangled states.

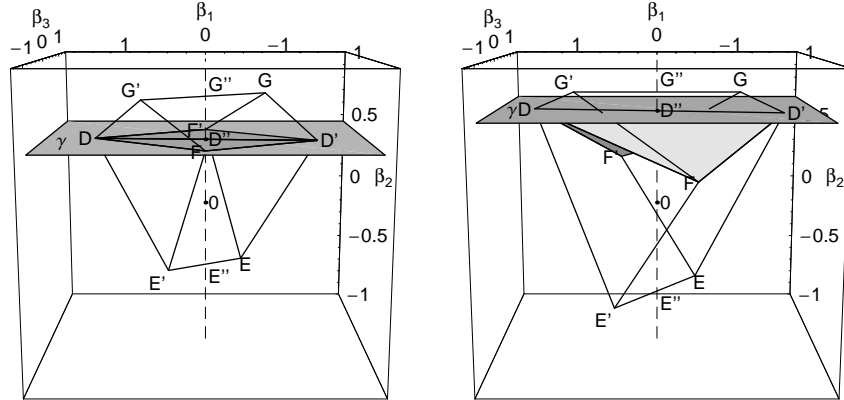


FIG. 2: The set of rotationally invariant PPT states for $j = 3/2$ and 3 , respectively. The γ plain labelled in the pictures is the boundary of the region in which entanglement is detected by Breuer's map. BE states which can be detected by the Breuer's map lie above the γ plain in the set of PPT operators.

Now we denote by γ the plain perpendicular to the line $E''G''$ and intersecting point D'' (Fig. 2). From relation (16) and a remark below it we can immediately conclude that all points lying in the PPT set above the γ plain are entangled.

Now we move to the points for which $\Phi_1(\varrho) \geq 0$ (such points lie on and below γ) and use the Π projection argument introduced in Sec II.B to show the separability of D and E (the separability of D' , E' results immediately from the properties of ϑ_1 map, namely preservation of separability).

Let us first recall the special symmetric case $4 \otimes 4$ solved in [10, 13]. In this case points D, D', F, F' lie on the γ plain and are all separable. The points E, E' are also separable and thus the γ plain is a boundary between the separable and bound entangled region.

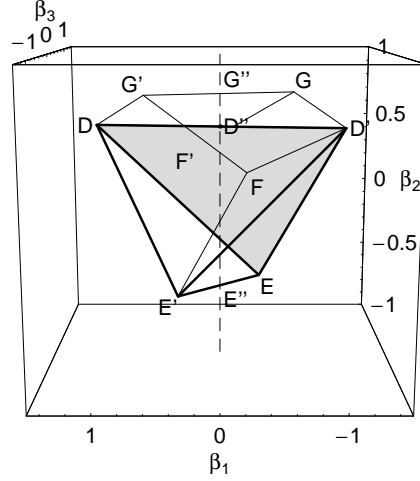


FIG. 3: The set of rotationally invariant PPT states for $j = 2$. The borders of the minimal separable region $DD'EE'$ are marked with thick lines.

In the general case $4 \otimes N$, the full characterization of the separable set is more difficult. Following the method developed by Breuer [14] we show that certain separable pure product states $|\phi^{(1)}\phi^{(2)}\rangle$ are projected by Π (see (12)) onto points D and E . For this purpose we take the functionals

$$\tilde{\beta}_K[\phi^{(1)}, \phi^{(2)}] = \sqrt{\frac{4N}{2K+1}} \sum_{q=-K}^K \langle \phi^{(1)} | T_{K,q}^{(1)} | \phi^{(1)} \rangle \langle \phi^{(2)} | T_{K,q}^{(2)\dagger} | \phi^{(2)} \rangle, \quad (22)$$

which, applying relation (12), map pure product states into the β space. Namely the $\tilde{\beta}_K$ functional is the β_K coordinate of a pure product state $|\phi^{(1)}\phi^{(2)}\rangle$ after action of Π projection. The matrix elements of $T_{K,q}$ used in the above equation are related to Wigner 3- j symbol by the formula [22]:

$$\langle j, m | T_{K,q} | j, m' \rangle = \sqrt{2K+1} (-1)^{j-m} \begin{pmatrix} j & j & K \\ m & m' & -q \end{pmatrix} \quad (23)$$

and by the relation $T_{K,q}^\dagger = (-1)^q T_{K,-q}$ one can determine the matrix elements of the conjugate.

To prove that E is separable it suffices to consider the states

$$|\tilde{\phi}^{(1)}\rangle = \left| \frac{3}{2}, -\frac{1}{2} \right\rangle, \quad |\tilde{\phi}^{(2)}\rangle = \left| \frac{N-1}{2}, \frac{N-1}{2} \right\rangle. \quad (24)$$

Due to the selection rules for 3- j symbol the only nonvanishing elements of the sum in (22) are those with $q = 0$, hence

$$\begin{aligned} \tilde{\beta}_1[\tilde{\phi}^{(1)}, \tilde{\phi}^{(2)}] &= \sqrt{\frac{4N}{3}} \langle \tilde{\phi}^{(1)} | T_{1,0}^{(1)} | \tilde{\phi}^{(1)} \rangle \langle \tilde{\phi}^{(2)} | T_{1,0}^{(2)\dagger} | \tilde{\phi}^{(2)} \rangle = -\sqrt{\frac{N-1}{5(N+1)}}, \\ \tilde{\beta}_2[\tilde{\phi}^{(1)}, \tilde{\phi}^{(2)}] &= \sqrt{\frac{4N}{5}} \langle \tilde{\phi}^{(1)} | T_{2,0}^{(1)} | \tilde{\phi}^{(1)} \rangle \langle \tilde{\phi}^{(2)} | T_{2,0}^{(2)\dagger} | \tilde{\phi}^{(2)} \rangle = -\sqrt{\frac{(N-1)(N-2)}{(N+1)(N+2)}}, \\ \tilde{\beta}_3[\tilde{\phi}^{(1)}, \tilde{\phi}^{(2)}] &= \sqrt{\frac{4N}{7}} \langle \tilde{\phi}^{(1)} | T_{3,0}^{(1)} | \tilde{\phi}^{(1)} \rangle \langle \tilde{\phi}^{(2)} | T_{3,0}^{(2)\dagger} | \tilde{\phi}^{(2)} \rangle = 3\sqrt{\frac{(N-1)(N-2)(N-3)}{5(N+1)(N+2)(N+3)}}. \end{aligned}$$

These are indeed the coordinates of E which implies that E represents the separable state.

To prove the separability of D we follow the arguments given by Breuer in [14]. Since D is an extreme point of the set of $SO(3)$ invariant states it has a single nonzero coordinate in α -space. The nonzero coordinate corresponds to the largest $J = J_{\max} = j_1 + j_2$. Consequently the spectral decomposition of the state always contains the projection on $|J_{\max}, J_{\max}\rangle$ which is a separable pure product state $|j_1, j_1\rangle \otimes |j_2, j_2\rangle$. This implies that for arbitrary j_1, j_2 one can find a separable state $|j_1, j_1\rangle \otimes |j_2, j_2\rangle$ which is mapped under Π to the state represented by point D .

The tetrahedron with vertices $DD'EE'$ in Fig. (3) represents the minimal separable set (see also [10]). The separability of the points F and F' as well as points lying between γ and the tetrahedron $DD'EE'$ within the PPT set is still undetermined. However in the case of $4 \otimes 5$ considered by Hendriks in [10] the set of separable states is given not only by points $DD'EE'$ but extends to the region near points F and F' . The separable states found numerically by Hendriks lie on lines $E'F$ and EF' . This result allows us to suspect that for $N \geq 5$ there also exist separable states outside the tetrahedron $DD'EE'$.

B. Bound entanglement in higher dimensional systems

Let us now move on to the general case of $\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ Hilbert space with even $N_1 \geq 4$ and arbitrary $N_2 \geq N_1$. For the purpose of proving the existence of BE in this state space we confine ourselves to the subset of ϑ_1 -invariant operators.

A set of rotationally and ϑ_1 -invariant states (\mathcal{R}_{inv}) can be easily determined in the β -space because the action of ϑ_1 in this representation is just a change of sign for β_K with odd K . As a result the states which remain unchanged after the partial time reversal must have coordinates indexed by odd K 's equal zero. Moreover, a normalized state has always $\beta_0 = 1$, so each rotationally and ϑ_1 -invariant state in $\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ with even N_1 is fully characterized by the set of $(N_1 - 2)/2$ parameters β_{2K} ($K = 1, 2, \dots, (N_1 - 2)/2$), i.e.,

$$\varrho_{\text{inv}} = (1, 0, \beta_2, 0, \beta_4, \dots, 0, \beta_{N_1-2}, 0). \quad (25)$$

To determine the range of the β_{2K} parameters for which vector β represents a density operator we impose the constraint of positivity. This can be easily done in P_J representation. Firstly we employ the $N_1 \times N_1$ matrix $L^{-1} = L^T$ (9) to find the α_J coordinates of a ϑ_1 -invariant state. This gives

$$\begin{aligned} \alpha_J = & (-1)^{\frac{N_1+N_2-2}{2}+J} \sqrt{2J+1} \left(\begin{Bmatrix} (N_1-1)/2 & (N_2-1)/2 & J \\ (N_2-1)/2 & (N_1-1)/2 & 0 \end{Bmatrix} \right. \\ & \left. + \sum_{K=1}^{(N_1-2)/2} \sqrt{4K+1} \begin{Bmatrix} (N_1-1)/2 & (N_2-1)/2 & J \\ (N_2-1)/2 & (N_1-1)/2 & 2K \end{Bmatrix} \beta_{2K} \right). \end{aligned} \quad (26)$$

The positivity condition $\alpha_J \geq 0$ leads to the set of inequalities for β_{2K} parameters. The β_{2K} parameters describing the $SO(3)$, ϑ_1 -invariant states lie between the hyperplanes given by equations $\alpha_J = 0$, where α_J 's are given by (26).

Our further reasoning follows from the structure of $4 \otimes N$ space, where Breuer's map detects the entanglement of ϑ_1 -invariant states above the D'' point.

The Breuer's map for rotationally, ϑ_1 -invariant states has the form presented in Ineq. (16). The identity matrix in β space is a vector with $\beta_0 = N_1 N_2$ and other coordinates equal zero. This follows from Eq. (8) and the fact that Q_0 operator is proportional to identity ($Q_0 = (N_1 N_2)^{-1/2} \mathbb{1}_{N_1} \otimes \mathbb{1}_{N_2}$). Therefore the β vector obtained after applying the Φ_1 map and normalization is:

$$\frac{2}{2 - N_1} \left(\frac{2 - N_1}{2}, 0, \beta_2, 0, \beta_4, \dots, 0, \beta_{N_1-2}, 0 \right). \quad (27)$$

One should notice that the vector is again ϑ_1 -invariant and that the Φ_1 map simply shifts the point with respect to point $(1, 0, \dots, 0)$.

We define the Γ hyperplane:

$$\Gamma : \frac{1}{\sqrt{N_1 N_2}} + (-1)^{N_2} \frac{2}{N_1 - 2} \sum_{K=1}^{(N_1-2)/2} \sqrt{4K+1} \begin{Bmatrix} (N_1-1)/2 & (N_2-1)/2 & (N_2-N_1)/2 \\ (N_2-1)/2 & (N_1-1)/2 & 2K \end{Bmatrix} \beta_{2K} = 0. \quad (28)$$

It crosscuts the PPT ϑ_1 -invariant set in such way that points lying on one side of it have always a negative eigenvalue after the action of map (16). To see this we apply the Φ_1 map to points lying on Γ . The obtained hyperplane is the boundary of \mathcal{R}_{inv} given by $\alpha_{(N_2-N_1)/2} = 0$. So points lying on one side of Γ are always mapped by Φ_1 onto points outside the set of positive operators, whereas points from the other side remain positive.

Now we prove that Γ always go through the interior of the \mathcal{R}_{inv} . To do it we find the analog of the point D'' from the case $4 \otimes N$. We denote it by \tilde{D}'' . It corresponds to the state $[\varrho_{\text{max}} + \vartheta_1(\varrho_{\text{max}})]/2$, where $\varrho_{\text{max}} = [N_1 N_2 (N_1 + N_2 - 1)]^{-1/2} P_{(N_1 + N_2 - 2)/2}$. \tilde{D}'' is certainly ϑ_1 -invariant since it has the form (14). It has all nonzero α_J 's, which implies that it belongs to the interior of \mathcal{R}_{inv} (recall that the boundaries of the set are given by $\alpha_J = 0$). The even (and thus nonzero) coordinates representing the \tilde{D}'' state in β space are:

$$\beta_{2K}^{(\tilde{D}'')} = \sqrt{N_1 N_2 (4K + 1)} (-1)^{N_2} \left\{ \begin{array}{ccc} (N_1 - 1)/2 & (N_2 - 1)/2 & (N_1 + N_2 - 2)/2 \\ (N_2 - 1)/2 & (N_1 - 1)/2 & 2K \end{array} \right\}, \quad (29)$$

with $K = 1, 2, \dots, (N_1 - 2)/2$. The coordinates fulfill the equality (28) (the proof is given in appendix A) so \tilde{D}'' always lies on the Γ plain. This implies that the Γ hyperplain cuts through the set of PPT ϑ_1 -invariant states. The above arguments prove the existence of BE states for all ϑ_1 -invariant states from $\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ Hilbert space with even N_1 and arbitrary $N_2 \geq N_1$. Moreover, by the relation (14) our arguments immediately extend to the set of PPT states.

We illustrate our previous considerations by an example from $6 \otimes N$ space (see Fig. 4). In this case the hyperplains defined by (26) become straight lines. One should notice that the BE region (gray area in Fig. 4) detected by the Breuer's map (3) shrinks with the growth of N .

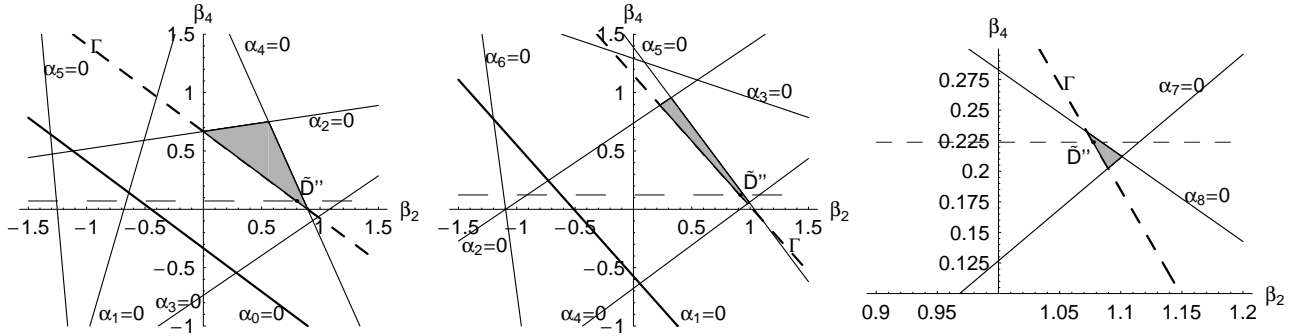


FIG. 4: The subset of rotationally and ϑ_1 -invariant states for dimensions $6 \otimes 6$, $6 \otimes 8$ and $6 \otimes 14$, respectively. The constant lines are the borders of the ϑ_1 -invariant set, namely $\alpha_J = 0$ ($J = (N_2 - 6)/2, \dots, (N_2 + 4)/2$), the thick constant line is $\alpha_{1/2(N_2-6)} = 0$, and the thick dashed line is the line Γ defined by Eq. (28). The gray area is the region in ϑ_1 -invariant set where bound entangled states are detected by the Breuer's map. The point where the dashed lines intersect is the point \tilde{D}'' .

IV. CONCLUSION

In the present paper we have shown that in $\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ Hilbert space of rotationally invariant states with even $N_1 \geq 4$ and arbitrary $N_2 \geq N_1$ there always exist bound entangled states among PPT states. This means simultaneously that partial transposition provides only a necessary criterion for separability in these cases. However, the problem is still unsolved for rotationally invariant states acting on $\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ with odd $N_1 \geq 5$ and arbitrary $N_2 \geq N_1$. Some preliminary results suggest that bound entanglement do exist in such systems. However, this issue requires further research.

We have described in more details the cases of $N_1 = 4$ and $N_1 = 6$. The provided examples give the geometrical description of the action of the map developed recently by Breuer [18] in the space of rotationally invariant states. Moreover they reveal that the BE PPT region detected by Breuer's map shrinks with the growth of asymmetry between the subsystems. Moreover, the example of $4 \otimes N$ shows that in this case the Breuer's map provides a necessary and sufficient criterion for $SO(3)$, ϑ_1 -invariant states. This suggests that for higher dimensional states of the form (25) the region of entanglement might be unambiguously determined with the criterion based on the Φ_1 map.

Let us discuss entanglement detection of the rotationally invariant states in the context of a recently proposed separability criterion involving determinant of the partially transposed density matrix [23]. Namely, it says that a given two-qubit state ρ is separable if and only if

$$\det \rho^{PT} \geq 0, \quad (30)$$

where PT denotes standard partial transposition with respect to an arbitrary subsystem. As shown in Ref. [11] and then in Ref. [14] partial transposition is also a sufficient criterion for separability for rotationally invariant states acting on $\mathbb{C}^2 \otimes \mathbb{C}^N$. Therefore one may expect that the above criterion applies also to this class of states. Using simple arguments it may be shown that this, indeed, is the case for even N .

Such states as well as their partial time reversal can be written in the form (5) and their eigenvalues are proportional to α_J 's. For such systems $\alpha_{N/2}$ of a state after partial time reversal (unitarily equivalent to partial transposition) is positive. This means that if a given rotationally invariant state is entangled its partial time reversal must have $\alpha_{(N-2)/2} < 0$, and therefore, in case of even N , odd number of negative eigenvalues (eigenvalues of $2 \otimes N$ rotationally invariant states with even N have odd degeneracy). of a density matrix after partial time reversal must be negative.

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APPENDIX A

In this appendix we show that point \tilde{D}'' (29) indeed belongs to the Γ hyperplane (28). To show this we rewrite Eq. (28) in the following form (here for simplicity we use $j_{1(2)}$ instead of $(N_{1(2)} - 1)/2$ whenever it leads to shorter formulas)

$$(-1)^{N_2} \sum_{K=0}^{(N_1-2)/2} \sqrt{4K+1} \left\{ \begin{matrix} j_1 & j_2 & j_2 - j_1 \\ j_2 & j_1 & 2K \end{matrix} \right\} \beta_{2K} + \frac{\sqrt{N_1 N_2}}{2N_2} = 0. \quad (\text{A1})$$

Now inserting coordinates of \tilde{D}'' given by Eq. (29) we obtain:

$$\sum_{K=0}^{(N_1-2)/2} (4K+1) \left\{ \begin{matrix} j_1 & j_2 & j_2 - j_1 \\ j_2 & j_1 & 2K \end{matrix} \right\} \left\{ \begin{matrix} j_1 & j_2 & j_1 + j_2 \\ j_2 & j_1 & 2K \end{matrix} \right\} + \frac{1}{2N_2} = 0. \quad (\text{A2})$$

We complete the sums on the left hand side with odd elements and receive:

$$\begin{aligned} & \sum_{K=0}^{N_1-1} (2K+1) \left\{ \begin{matrix} j_1 & j_2 & j_2 - j_1 \\ j_2 & j_1 & K \end{matrix} \right\} \left\{ \begin{matrix} j_1 & j_2 & j_1 + j_2 \\ j_2 & j_1 & K \end{matrix} \right\} + \sum_{K=0}^{N_1-1} (-1)^K (2K+1) \left\{ \begin{matrix} j_1 & j_2 & j_2 - j_1 \\ j_2 & j_1 & K \end{matrix} \right\} \left\{ \begin{matrix} j_1 & j_2 & j_1 + j_2 \\ j_2 & j_1 & K \end{matrix} \right\} \\ & = -1/N_2. \end{aligned} \quad (\text{A3})$$

Both sums on the left-hand side of the above equation can be calculated with the help of the following relations [22]

$$\sum_K (2J+1)(2K+1) \left\{ \begin{matrix} a & b & J \\ c & d & K \end{matrix} \right\} \left\{ \begin{matrix} a & b & J' \\ c & d & K \end{matrix} \right\} = \delta_{JJ'} \quad (\text{A4})$$

and

$$\sum_K (-1)^K (2K+1) \left\{ \begin{matrix} a & b & J \\ c & d & K \end{matrix} \right\} \left\{ \begin{matrix} a & c & J' \\ b & d & K \end{matrix} \right\} = (-1)^{J+J'} \left\{ \begin{matrix} a & b & J \\ d & c & J' \end{matrix} \right\}. \quad (\text{A5})$$

The first sum in Eq. (A3) equals 0 by the orthogonality relation (A4), and the second sum reduces to:

$$\sum_{K=0}^{N_1-1} (-1)^K (2K+1) \left\{ \begin{matrix} j_1 & j_2 & j_2 - j_1 \\ j_2 & j_1 & K \end{matrix} \right\} \left\{ \begin{matrix} j_1 & j_2 & j_1 + j_2 \\ j_2 & j_1 & K \end{matrix} \right\} = (-1)^{2j_2} \left\{ \begin{matrix} j_1 & j_2 & j_2 - j_1 \\ j_1 & j_2 & j_1 + j_2 \end{matrix} \right\} = -\frac{1}{2j_2+1} = -\frac{1}{N_2}, \quad (\text{A6})$$

which is the right hand side of Eq. (A3). ■

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